Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

Nonlinear Control and Reduction of Underactuated Systems with Symmetry III: Input Coupling Case

Reza Olfati-Saber
California Institute of Technology
Control and Dynamical Systems 107-81
Pasadena, CA 91125
olfati@cds.caltech.edu

Abstract

In this paper, we address nonlinear control and reduction of high-order underactuated mechanical systems with kinetic symmetry and input coupling. In some aerospace vehicles, the effects of the control of the attitude dynamics appears in the translational dynamics. We present a general framework for decoupling of these effects. The decoupling is done by applying a change of coordinates in explicit form that transforms the original system into a cascade nonlinear system. We obtain three types of cascade normal forms, namely, nonlinear systems in strict feedback form, strict feedforward form, and nontriangular quadratic form. For a class of underactuated systems that are differentially flat, we obtain the flat outputs automatically as a by-product of the decoupling change of coordinates.

1 Introduction

Control design and analysis for underactuated mechanical systems is currently an active field of research. The importance of underactuated systems is due to their broad applications in Robotics (e.g. mobile robots, flexible-link robots, snake-type robots, walking robots), Aerospace systems (e.g. aircraft, spacecraft, helicopters, and satellites), and Marine vehicles (e.g. surface vessels and underwater vehicles). In addition, restriction of the control authority in underactuated systems offers challenging control problems from theoretical point of view (see [7, 10] for recent surveys).

This paper is part III of a series of articles that aim to address reduction and nonlinear control of broad classes of high-order underactuated systems. In [7], it is shown that underactuated systems can be essentially classified to eight main classes that overall cover the majority of the aforementioned real-life applications. In part III, we focus on control of a class of underactuated systems with input coupling referred to as Class-IV systems. In section 2, we precisely define this class of underactuated systems.

The main contribution of part III of this paper is providing change of coordinates in closed-form that transform high-order underactuated systems with input coupling into cascade systems. There are three different types of normal forms that are obtained; namely, strict feedback form, strict feedforward form, and nontriangular quadratic form. The nonlinear systems in strict feedback form can be effectively stabilized using backstepping procedure [3, 6]. Also, systems in feedforward forms can be stabilized using Teel's nested saturations [12], and feedforwarding methods due to Mazenc and Praly [5]. However, stabilization of nonlinear systems in nontriangular form is in general an open problem. In important special cases, this has been addressed in [7, 8].

The outline of the paper is as follows. In section 2, we provide some background on dynamics and symmetry properties of underactuated systems. In section 3, we state our main reduction and stabilization results. In section 4, we present the VTOL aircraft as an example. Finally, we make concluding remarks.

2 Underactuated Systems with Symmetry

In this paper, we consider the class of simple Lagrangian systems with configuration vector $q = \mathrm{col}(q_x, q_s) \in Q_x \times Q_s$, configuration space $Q = Q_x \times Q_s$ of dimension n, and Lagrangian

$$\mathcal{L}(q, \dot{q}) = K - V = \frac{1}{2}\dot{q}^T M(q_s)\dot{q} - V(q_x, q_s) \tag{1}$$

where K is the kinetic energy, $V(q)$ is the potential energy, and $M(q) = M(q_s)$ is the *inertia matrix*. We say the system has *kinetic symmetry* w.r.t. q_x due to $\frac{\partial K}{\partial q_x} = 0$. We refer to q_x and q_s as the vectors of *external variables* and *shape variables*, respectively. The forced Euler-Lagrange equation for a system with Lagrangian (1) can be written as

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_x} - \frac{\partial \mathcal{L}}{\partial q} &= F_x(q)\tau \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_s} - \frac{\partial \mathcal{L}}{\partial q} &= F_s(q)\tau \end{aligned} \tag{2}$$

where $\tau \in \mathbb{R}^m$ and $F(q) = \mathrm{col}[F_x(q), F_s(q)]$ is the *force matrix*. Throughout the paper, we assume the following conditions hold:

i) $m = \mathrm{rank}(F(q)) < n$ (i.e. the system is underactuated)

ii) $F_x(q) \not\equiv 0$

iii) $F_s(q)$ is a full rank matrix.

Due to conditions ii) and iii), we say the system possesses *input coupling* property w.r.t. the control input τ.

Remark 1. This input coupling effect is particularly important in aerospace applications because such a coupling naturally exists in an accurate model of an aircraft or a helicopter (with both 6 DOF and 3 DOF).

We make the following further simplifying assumptions:

Assumption 1. The inertia matrix $M(q_s)$ is a block diagonal matrix, i.e. $M(q_s) = \mathrm{diag}(m_{xx}(q_s), m_{ss}(q_s))$.

Assumption 2. Assume $F(q) = F(q_s)$, the force matrix does not depend on the external variables .

Under Assumptions 1 and 2, the Lagrangian equations of motion in (2) can be written as

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + h_x(q, \dot{q}) &= F_x(q_s)\tau \\ m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) &= F_s(q_s)\tau \end{aligned} \tag{3}$$

where h_x, h_s contain the Coriolis, centrifugal, and gravity terms. The dynamics of (3) can be partially linearize using applying the following global change of control

$$\tau = F_s^{-1}(q_s)(m_{ss}(q_s)u + h_s(q, \dot{q})) \tag{4}$$

that reduces the dynamics of the shape variables to $\ddot{q}_s = u$.

We find the following quadratic forms convenient for representation of the normal forms of underactuated systems with input coupling.

Definition 1. (Vector Quadratic Forms) Consider a mapping $\Sigma : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Sigma(x,v) = v^T \Pi(x) v := (v^T \pi_1 v, \dots, v^T \pi_n v)^T \tag{5}$$

where $\Pi(x) : \mathbb{R}^p \to \mathbb{R}^{n\times n\times n}$ is a cubic matrix with layers $\pi_i(x) : \mathbb{R}^p \to \mathbb{R}^{n\times n}, i = 1, \dots, n$ which are square matrices. We call $\Sigma(x,v)$ a *vector quadratic form* in v.

Remark 2. For the definitions of *reduction* and *integrability* of momentum terms, please refer to parts I and II of this paper.

3 Main Results

We begin with reduction of the general case of non-flat underactuated mechanical systems with kinetic symmetry and input coupling. By a non-flat mechanical system, we mean that the inertia matrix is not constant (this should not be mistaken with the notion of "differential flatness", though they are closely related). Here is our first main result that leads to a nontriangular normal form:

Theorem 1. *Consider the underactuated system in (3) with input coupling and fully-actuated shape variables. Suppose all the elements of*

$$\omega = m_{xx}^{-1}(q_s) F_x(q_s) F_s^{-1}(q_s) m_{ss}(q_s) dq_s$$

are exact one-forms and let $\omega = d\gamma(q_s)$. Then, the following global change of coordinates (i.e. diffeomorphism)

$$\begin{aligned} q_r &= q_x - \gamma(q_s) \\ p_r &= m_{xx}(q_s)p_x - F_x(q_s)F_s^{-1}(q_s)m_{ss}(q_s)p_s \end{aligned} \tag{6}$$

with $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$ transforms (3) into the following cascade system in nontriangular quadratic normal form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \tag{7}$$

In addition, the (q_r, p_r)-subsystem is a Lagrangian system with configuration vector q_r and parametrized reduced Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2}\dot{q}_r^T m_r(q_s)\dot{q}_r - V_r(q_r, q_s)$$

that satisfies the following forced Euler-Lagrange equation

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} &= \Sigma(q_s, p_r, p_s) \\ &- F_x(q_s)F_s^{-1}(q_s)g_s(q_r + \gamma(q_s), q_s) \end{aligned}$$

where

$$\begin{aligned} m_r(q_s) &:= m_{xx}(q_s) \\ V_r(q_r, q_s) &:= V(q_r + \gamma(q_s), q_s) \\ g_r(q_r, q_s) &:= g_x(q_x, q_s) - F_x(q_s)F_s^{-1}(q_s)g_s(q_x, q_s) \\ g_x(q_x, q_s) &:= -\nabla_{q_x} V(q_x, q_s) \\ g_s(q_x, q_s) &:= -\nabla_{q_s} V(q_x, q_s) \end{aligned}$$

and the vector quadratic form Σ is given by

$$\Sigma(q_s, p_r, p_s) = \begin{bmatrix} p_r \\ p_s \end{bmatrix}^T \Pi(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

is a quadratic form in (p_r, p_s) with a cubic weight matrix $\Pi(q_s)$.

Proof. See pages 77–79 in [7]. □

The following result provides classes of underactuated systems in Theorem 1 that can be globally transformed into nonlinear systems in feedforward form. This makes it possible to apply the existing feedforwarding results to control of underactuated systems.

Theorem 2. *Assume all the conditions in Theorem 1 hold. In addition, suppose* *i) $m_r = m_{xx}$ is constant.*

ii) $V(q) = k_0^T q_x + V(q_s)$ where k_0 is a constant vector.

Then, the global change of coordinates in (6) transforms the dynamics of the underactuated system in (3) into a cascade nonlinear system in feedforward form as the following

$$\begin{array}{rcl} \dot{q}_r &=& m_r^{-1} p_r \\ \dot{p}_r &=& g_r(q_s) + p_s^T \Pi(q_s) p_s \\ \dot{q}_s &=& p_s \\ \dot{p}_s &=& u \end{array} \tag{8}$$

where $\Pi(q_s)$ is a cubic matrix and

$$g_r(q_s) = -k_0 + F_x(q_s) F_s^{-1}(q_s) \nabla_{q_s} V(q_s)$$

Moreover, if the following conditions are satisfied

iii) $dim(q_x) = dim(q_s)$.

iv) $g_r(0) = 0$ and the Jacobian matrix $\nabla_{q_s} g_r(q_s)$ is invertible at $q_s = 0$.

Then, the origin $(q_r, p_r, q_s, p_s) = 0$ for the nonlinear system in (8) can be globally asymptotically stabilized using a state feedback in explicit form as nested saturations.

Proof. See page 80 in [7]. □

Now, we present our final result on reduction of flat underactuated systems with input coupling. In this case, we add another input to the dynamics of the external variables in the form $F_r(q_s)\tau_r$. This extra input represents the body thrust in a helicopter or an aircraft.

Theorem 3. *Consider the following flat underactuated system satisfying $V(q) = V(q_x)$*

$$\begin{array}{rcl} m_{xx}\ddot{q}_x - g_x(q_x) &=& F_r(q_s)\tau_r + F_x(q_s)\tau \\ m_{ss}\ddot{q}_s &=& F_s(q_s)\tau \end{array} \tag{9}$$

where $g_x(q_x) = -\nabla_{q_x} V(q_x)$, $\tau_r \in \mathbb{R}$, $F_r(q_s) : \mathbb{R}^m \to \mathbb{R}^{n-m}$ is a unit vector that is onto over a unit ball in $\mathbb{R}^{n-m}$, $\tau \in \mathbb{R}^m$, m_{xx} and m_{ss} are constant, and $F_s(q_s)$ is an $m \times m$ invertible matrix. Let $\tau = F_s(q_s)^{-1} m_{ss} u$. Assume all the elements of

$$\omega = m_{xx}^{-1} F_x(q_s) F_s^{-1}(q_s) m_{ss} dq_s$$

are exact one-forms and let $\omega = d\gamma(q_s)$. Then, the following global change of coordinates

$$\begin{array}{rcl} q_r &=& q_x - \gamma(q_s) \\ p_r &=& m_{xx} p_x - F_x(q_s) F_s^{-1}(q_s) m_{ss} p_s \end{array} \tag{10}$$

with $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$ transforms the dynamics of (9) into the following form

$$\begin{array}{rcl} \dot{q}_r &=& m_r^{-1} p_r \\ \dot{p}_r &=& g_r(q_r + \gamma(q_s)) - p_s^T \pi_F(q_s) p_s + F_r(q_s)\tau_r \\ \dot{q}_s &=& p_s \\ \dot{p}_s &=& u \end{array} \tag{11}$$

where $\pi_F(q_s)$ is a cubic matrix. In addition, if the following conditions hold:

i) $p_s^T \pi_F(q_s) p_s = (p_s^T Q p_s) F_r(q_s)$ with $Q \in \mathbb{R}^{m^2}$.

ii) $V(q_x) = k_0^T q_x$ where $k_0 \in \mathbb{R}^{n-m}$ is a constant.

Then, the reduced Lagrangian system is a fully-actuated flat system that satisfies

$$\frac{d}{dt}\frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = F_r(q_s)\tilde{\tau}_r \tag{12}$$

where

$$\tilde{\tau}_r = \tau_r - p_s^T Q p_s$$

is the new scalar control.

Proof. The proof is by direct calculation. □

4 Example: The VTOL Aircraft

In this section, we present a detailed reduction process for the VTOL (vertical take off and landing) aircraft (see [9] for control design and simulation results). The simplified dynamics of the VTOL aircraft, as shown in Figure 1, is given in [2, 4] as the following

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -u_1\sin(\theta) + \epsilon u_2\cos(\theta) \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= u_1\cos(\theta) + \epsilon u_2 \sin(\theta) - g \\ \dot{\theta} &= \omega \\ \dot{\omega} &= u_2 \end{aligned} \tag{13}$$

where $\epsilon \neq 0$. Here, θ denotes the roll angle and the plane moves in a vertical (x_1, y_1)-plane. Clearly, the system has three degrees

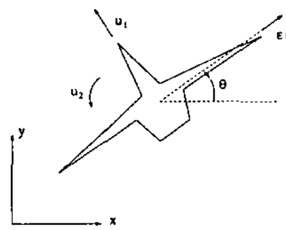

Figure 1: The VTOL aircraft.

of freedom and only two actuators–thus it is underactuated. The effect of the body torque u_2 appears in the translational dynamics of the VTOL aircraft by a factor of ϵ. This captures a general property of real aircrafts with 6 DOF that exhibit a similar input coupling effect [2]. The case with $|\epsilon| \ll 1$ resulting in a weak input coupling has been mostly considered in the literature [2, 4, 11]. Here, we are interested in the strong input coupling case with an arbitrary $\epsilon \neq 0$. To eliminate the effect of u_2 in the translational dynamics of the VTOL aircraft, first let us rewrite the dynamics of the VTOL as

$$\begin{aligned} m_{xx}\ddot{q}_x - g_x(q_x) &= F_r(q_s)\tau_r + F_x(q_s)\tau \\ m_{ss}\ddot{q}_s &= F_s(q_s)\tau \end{aligned} \tag{14}$$

where $q_x = (x_1, y_1)^T$, $q_s = \theta$, $m_{xx} = I_{2\times 2}$, $m_{ss} = 1$, $F_s = 1$, and F_r, F_x, g_x are given by

$$F_r(q_s) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}, \; F_x(q_s) = \begin{bmatrix} \epsilon\cos(\theta) \\ \epsilon\sin(\theta) \end{bmatrix}$$
$$g_x = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

which clearly shows $F_r(q_s)$ is a unit vector. Based on theorem 3, we need to check whether the vector of one-forms ω defined as $\omega = m_{xx}^{-1}F_x(q_s)F_s^{-1}(q_s)m_{ss}dq_s$ has exact elements or not. By direct calculation, we get

$$\omega = F_x(q_s)dq_s = \begin{bmatrix} \epsilon\cos(\theta)d\theta \\ \epsilon\sin(\theta)d\theta \end{bmatrix}$$

which has exact elements. Moreover, ω has an exact differential $\gamma(\theta) = [\epsilon\sin(\theta), -\epsilon(\cos(\theta) - 1)]^T$ that satisfies $\omega = d\gamma(\theta)$. Therefore, applying the following global change of coordinates

$$\begin{aligned} z_1 &= x_1 - \epsilon\sin(\theta) \\ z_2 &= x_2 - \epsilon\cos(\theta)\omega \\ w_1 &= y_1 + \epsilon(\cos(\theta) - 1) \\ w_2 &= y_2 - \epsilon\sin(\theta)\omega \\ \xi_1 &= \theta \\ \xi_2 &= \omega \end{aligned} \tag{15}$$

transforms the dynamics of the VTOL aircraft into

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\sin(\xi_1)\tilde{u}_1 =: v_1 \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= \cos(\xi_1)\tilde{u}_1 - g =: v_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u_2 \end{aligned} \tag{16}$$

and decouples the translational and attitude dynamics. Here, $\tilde{u}_1 = u_1 - \epsilon\xi_2^2$ is the new control. Clearly, (z_1, w_1) are two flat outputs of the VTOL aircraft which are obtained automatically as the by-products of the decoupling change of coordinates (see [1, 7, 13, 14] for more details on differential flatness and flat outputs).

5 Conclusion

We addressed nonlinear control and reduction of high-order underactuated mechanical systems with kinetic symmetry and input coupling. This is particularly important in stabilization/tracking problems for aerospace vehicles where the effects of the control of the attitude dynamics appears in the translational dynamics. We presented a method for obtaining a decoupling change of coordinates that transforms the dynamics of the original underactuated system into nonlinear systems in strict feedback form, strict feedforward form, and nontriangular quadratic form. This allows application of the existing nonlinear control design methods known as backstepping/feedforwarding techniques to control of high-order underactuated systems. For a special class of underactuated systems that are differentially flat, the flat outputs are obtained automatically as a by-product of the decoupling change of coordinates.

References

[1] M. Fliess, J. Levine, P. Martin, and P. Rouchon. "Flatness and defect of nonlinear systems: Introductory theory and examples.". *Int. Journal of Control*, 61(6):1327–1361, 1995.

[2] J. Hauser, S. Sastry, and G. Meyer. "Nonlinear control design for slightly nonminimum phase systems". *Automatica*, vol. 28:pp.665–679, 1992.

[3] A. Isidori. *Nonlinear Control Systems.* Springer, 1995.

[4] P. Martin, S. Devasia, and B. Paden. "A different look at output tracking: control of a VTOL aircraft". *Proceedings of 33rd IEEE Conference on Decision and Control*, pages 2376–2381, 1994.

[5] F. Mazenc and L. Praly. "Adding integrations, saturated controls, and stabilization for feedforward Systems". *IEEE Trans. on Automatic Control*, 40:1559–1578, 1996.

[6] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design.* John Wiley & Sons, 1995.

[7] R. Olfati-Saber. *"Nonlinear Control of Underactuated Mechanical Systems with Application to Robotics and Aerospace Vehicles".* PhD thesis, Massachusetts Institute of Technology, Department of Electrical Engineering and Computer Science, February 2001.

[8] R. Olfati-Saber. "Fixed point controllers and stabilization of the cart-pole system and the rotating pendulum". *Proc. of the 38th Conf. on Decision and Control*, 2:1174–1181, Phoenix, AZ, Dec. 1999.

[9] R. Olfati-Saber. "Global configuration stabilization for the VTOL aircraft with strong input coupling". *Proc. of the 39th Conf. on Decision and Control*, 4:3588–3589, Sydney, Australia, Dec. 2000.

[10] M. Reyhanoglu, A. van der Schaft, N. H. McClamroch, and I. Kolmanovsky. "Dynamics and control of a class of underactuated mechanical systems". *IEEE Trans. on Automatic Control*, 44(9):1663–1671, 1999.

[11] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonlinear Control.* Springer-Verlag, 1997.

[12] A. R. Teel. "Using Saturation to stabilize a class of single-input partially linear composite systems". *IFAC NOLCOS'92 Symposium*, pages 369–374, June 1992.

[13] M. van Nieuwstadt and R. Murray. "Real-time trajectory generation for differentially flat systems". *Int. Journal of Robust and Nonlinear Control*, 8(11):995–1020, 1998.

[14] M. van Nieuwstadt and R. Murray. "Approximate trajectory generation for differentially flat systems with zero dynamics". *Proc. of the 34th Conf. on Decision and Control*, New Orleans, LA, Dec. 1995.